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Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness

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Abstract

We study spacetimes with a closed conformal Killing-Yano tensor. It is shown that the D-dimensional Kerr-NUT-de Sitter spacetime constructed by Chen-Lü-Pope is the only spacetime admitting a rank-2 closed conformal Killing-Yano tensor with a certain symmetry.

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Higher dimensional black hole solutions have attracted renewed interests in the recent developments of supergravity and superstring theories. Recently, the D-dimensional Kerr-NUT-de Sitter metrics were constructed by [1]. All the known vacuum type D black hole solutions are included in these metrics [2]. Kerr-NUT-de Sitter metrics are also interesting from the point of view of AdS/CFT correspondence. Indeed, odd-dimensional metrics lead to Sasaki-Einstein metrics by taking BPS limit [1, 3, 4, 5] and even-dimensional metrics lead to Calabi-Yau metrics in the limit [1, 6, 7]. Especially, the five-dimensional Sasaki-Einstein metrics have emerged quite naturally in the AdS/CFT correspondence.

On the other hand, it has been shown that geodesic motion in the Kerr-NUT-de Sitter spacetime is integrable for all dimensions [8, 9, 10, 11, 12, 13]. Indeed, the constants of motion that are in involution can be explicitly constructed from a rank-2 closed conformal Killing-Yano (CKY) tensor. In this paper, using a geometric characterisation of the separation of variables in the Hamilton-Jacobi equation [14], we study spacetimes with a rank-2 closed CKY tensor.

The rank-2 CKY tensor is defined as a two-form

$$h = \frac{1}{2} h_{ab}\, \mathrm{d}x^a \wedge \mathrm{d}x^b, \qquad h_{ab} = -h_{ba} \tag{1}$$

satisfying the equation [15]

$$\nabla_a h_{bc} + \nabla_b h_{ac} = 2\xi_c g_{ab} - \xi_a g_{bc} - \xi_b g_{ac}. \tag{2}$$

The vector field ξ_a is called the associated vector of h_{ab}, which is given by

$$\xi_a = \frac{1}{D-1} \nabla^b h_{ba}. \tag{3}$$

Theorem *Let us assume the existence of a single rank-2 CKY tensor h for D-dimensional spacetime (M, g) satisfying the conditions,*

$$(a)\ dh = 0, \quad (b)\ \mathcal{L}_\xi g = 0, \quad (c)\ \mathcal{L}_\xi h = 0. \tag{4}$$

Then, M is only the Kerr-NUT-de Sitter spacetime[1].

The Kerr-NUT-de Sitter metric takes the form [1]:
(a) $D = 2n$

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left(\sum_{j=0}^{n-1} A_\mu^{(j)} d\psi^j \right)^2 . \tag{5}$$

[1]We require a further technical condition which will be detailed in the proof. See the assumption below eq.(14).

(b) $D = 2n+1$

$$g = \sum_{\mu=1}^{n} \frac{dx_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n} Q_\mu \left(\sum_{j=0}^{n-1} A_\mu^{(j)} d\psi^j \right)^2 + S \left(\sum_{j=0}^{n} A^{(j)} d\psi^j \right)^2 . \tag{6}$$

The functions Q_μ are given by

$$Q_\mu = \frac{X_\mu}{U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^{n} (x_\mu^2 - x_\nu^2), \tag{7}$$

where X_μ is an arbitrary function depending only on ${x_\mu}^2$ and

$$A_\mu^{(k)} = \sum_{\substack{1 \le \nu_1 < \cdots < \nu_k \le n \\ (\nu_i \neq \mu)}} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \quad A^{(k)} = \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \tag{8}$$

($A_\mu^{(0)} = A^{(0)} = 1$) and $S = c/A^{(n)}$ with a constant c.

In the following we briefly describe the proof (see [16] for detailed analysis). The wedge product of two CKY tensors is again a CKY tensor and so the wedge powers $h^{(j)} = h \wedge \cdots \wedge h$ are CKY tensors. The condition (a) means that the Hodge dual $(D-2j)$-forms $f^{(j)} = *h^{(j)}$ are Killing-Yano tensors:

$$\nabla_{(a_1} f^{(j)}_{a_2) a_3 \cdots a_{D-2j+1}} = 0. \tag{9}$$

These Killing-Yano tensors generate the rank-2 Killing tensors $K^{(j)}$ obeying the equation $\nabla_{(a} K^{(j)}_{bc)} = 0$. Under the condition (a) the Killing tensors $K^{(j)}$ are mutually commuting [12, 13],

$$[K^{(i)},\ K^{(j)}]_S = 0. \tag{10}$$

The bracket $[\quad,\quad]_S$ represents a symmetric Schouten product. The equation can be written as

$$K^{(i)}_{d(a} \nabla^d K^{(j)}_{bc)} - K^{(j)}_{d(a} \nabla^d K^{(i)}_{bc)} = 0. \tag{11}$$

Let us define the vector fields $\eta^{(j)}$ by

$$\eta_a^{(j)} = K^{(j)}{}_a{}^b \xi_b. \tag{12}$$

[2] We call the metric Kerr-NUT-de Sitter for an arbitrary X_μ. The existence of h does not restrict the metric to be Einstein.

Then we have

$$\nabla_{(a}\eta^{(j)}_{b)} = \frac{1}{2}\mathcal{L}_\xi K^{(j)}_{ab} - \nabla_\xi K^{(j)}_{ab}, \tag{13}$$

which vanishes by the conditions (b) and (c), i.e. $\eta^{(j)}$ are Killing vectors. We can show that Killing vectors $\eta^{(i)}$ and Killing tensors $K^{(j)}$ are mutually commuting [14],

$$[\eta^{(i)},\ K^{(j)}]_S = 0, \quad [\eta^{(i)},\ \eta^{(j)}] = 0. \tag{14}$$

Here, we assume that the Killing tensors $K^{(j)}$ and $K^{(ij)} = \eta^{(i)} \otimes \eta^{(j)} + \eta^{(j)} \otimes \eta^{(i)}$ are independent. Therefore all the separability conditions of the geodesic Hamilton-Jacobi equation are satisfied [14].

Let y^a be geodesic separable coordinates of $D = (n+k)$-dimensional spacetime M:

$$y^a = (x^\mu,\ \psi^i), \quad \mu = 1, 2, \cdots, n, \quad i = 0, 1, \cdots, k-1, \tag{15}$$

where $k = n$ ($k = n+1$) for D even (odd). In these coordinates the commuting Killing vectors $\eta^{(j)}$ ($j = 0, 1, \cdots, k-1$) are written as $\eta^{(j)} = \partial/\partial\psi^j$. From [17, 18, 19] the inverse metric components are of the form,

$$g^{\mu\mu} = \bar{\phi}^\mu_{(0)}(x), \quad g^{ij} = \sum_{\mu=1}^{n} \zeta^{ij}_\mu(x^\mu)\bar{\phi}^\mu_{(0)}(x), \tag{16}$$

and the components of the Killing tensors $K^{(j)}$ are given by

$$K^{(j)\mu\nu} = \delta^{\mu\nu}\bar{\phi}^\mu_{(j)}(x), \quad K^{(j)\mu i} = 0, \quad K^{(j)i\ell} = \sum_{\mu=1}^{n} \zeta^{i\ell}_\mu(x^\mu)\bar{\phi}^\mu_{(j)}(x). \tag{17}$$

Here, $\bar{\phi}^\mu_{(j)}$ is the j-th column of the inverse of an $n \times n$ Stäckel matrix $(\phi^{(j)}_\mu)$, i.e. each element depends on the variable corresponding to the lower index only: $\phi^{(j)}_\mu(x^\mu)$. It should be noticed that the Killing tensors are constructed from CKY tensors, so that they obey the following recursion relations as linear operators [14]:

$$K^{(j)} = A^{(j)}I - QK^{(j-1)}, \tag{18}$$

where I is an identity operator and Q is defined by

$$Q^a{}_b = -h^a{}_c h^c{}_b. \tag{19}$$

Here $A^{(j)}$ is given by

$$\det{}^{1/2}(I + \beta Q) = \sum_{j=0}^{n} A^{(j)}\beta^j. \tag{20}$$

Note that the equation (2) with the condition (a) is equivalent to

$$\nabla_a h_{bc} = \xi_c g_{ab} - \xi_b g_{ac}. \tag{21}$$

We can further restrict the unknown functions $\bar{\phi}^{\mu}_{(0)}$ and ζ^{ij}_{μ} in the metric (16). This is analyzed by considering the equation (21) with $\xi = \eta^{(0)}$, and finally we find the Kerr-NUT-de Sitter metric (5) or (6).

As a crosscheck of our theorem, we confirmed by the direct calculation that a CKY tensor satisfying (a), (b) and (c) does not exist in the five-dimensional black ring background [20].

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